Linear Algebra and its Applications 431 (2009) 2142–2153

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Solutions to operator equations on Hilbert C^*-modules ☆

Xiaochun Fang*, Jing Yu, Hongliang Yao

Department of Mathematics, Tongji University, Shanghai 200092, China

A R T I C L E I N F O

Article history:
Received 14 May 2008
Accepted 5 July 2009
Available online 7 August 2009

Submitted by C.K. Li

AMS classification:
46L08
47A05

Keywords:
Hilbert C^*-module
Solution
Hermitian solution
Positive solution

A B S T R A C T

Let $\mathcal{A}$ be a C^*-algebra, E, F and G be Hilbert $\mathcal{A}$-modules, $T \in L_{\mathcal{A}}(E, F)$, and $T' \in L_{\mathcal{A}}(G, F)$. We generalize the Douglas theorem about the operator equation $TX = T'$ from Hilbert space to Hilbert C^*-module. To the equation $TX = T'$ and to the system of two equations $TX = T'$ and $XS = S'$, we get the forms of general solutions (in the case that there exists a solution), and give some sufficient and necessary conditions for the existence of solutions, and the existence of hermitian solutions and positive solutions (in the case $G = E$). In addition, the forms of general hermitian solution and general positive solution (in the case that there exists a solution and $G = E$) to the equation $TX = T'$ are given too.

1. Introduction and preliminary

Much progress has been made on the study of operator equations for operators on a Hilbert space. In 1966, Douglas studied the equation $AX = B$ for operators on a Hilbert space, and gave the famous Douglas theorem in [3], which was stated as follows:

Let H be a Hilbert space, $A, B \in B(H)$. The following statements are equivalent:

(1) $R(A) \subset R(B)$;
(2) $AA^* \leqslant \lambda^2 BB^*$ for some $\lambda \geqslant 0$; and
(3) There exists a bounded operator $C \in B(H)$ such that $A = BC$.

☆ This article is supported by the National Natural Science Foundation of China (10771161).

* Corresponding author.
E-mail addresses: xfang@tongji.edu.cn (X. Fang), yj19800830@163.com (J. Yu).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.07.009

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator C such that

(i) $\|C\|^2 = inf\{\mu : AA^* \leqslant \mu BB^*\}$;
(ii) $N(A) = N(C)$; and
(iii) $R(C) \subset \overline{R(B^*)}$.

Where $R(A), N(A)$ denote the rang and null space, respectively.

Another way to study the solutions to operator equations for operators on a Hilbert space (or for operators in a C^*-algebra) is by use of the generalized inverses of operators (for example see [1,2]). In [2], Dajić and Koliha got the characterization for the existence of common hermitian and positive solutions to the equations $AX = C, XB = D$ for operators on a Hilbert space.

Hilbert C^*-module is a natural generalization both of Hilbert space and of C^*-algebra, and it has been an important tool in the theory of C^*-algebra, especially in the study of KK-groups and induced representations (see [4–6,8,9,11]). Therefore it is meaningful to put forward a generalized version of the previous results about operator equations in the context of Hilbert C^*-modules.

In [13], by use of the generalized inverses of adjointable operators on a Hilbert C^*-module (for example see [14]), and along the same line as in [2], Q. Xu gave the concrete representations of common hermitian and positive solutions to the equations $AX = C, XB = D$ for operators on a Hilbert C^*-module. In addition, one equivalent condition for the existence of solutions to the equation $AX = C$ was given, which could be stated as follows:

Let E, F be two Hilbert $\mathcal{A}$-modules, $A \in L_{\mathcal{A}}(E, F)$ with closed range and $C \in L_{\mathcal{A}}(E, F)$. Then $AX = C$ has a solution $X \in L_{\mathcal{A}}(E)$ if and only if $R(C) \subseteq R(A)$. In which case, the general solution is of the form

$$X = A^-C + (I_E - A^-A)T,$$

where A^- is an inner inverse of A and $T \in L_{\mathcal{A}}(E)$ is arbitrary.

In [2,13], to use the generalized inverses, the authors have to restrict their attentions to those adjointable operators whose ranges are closed (even in Hilbert space case). This leads us to study the solutions to the equation $AX = C$ and to the system of equations $AX = C, XB = D$ for more general adjointable operators on a Hilbert C^*-module without the assumption of closed ranges. Fortunately the Douglas theorem above gives us the hint. In this paper, we restrict our attentions to the operators the closures of the ranges of whose adjoint operators are orthogonally complemented. It should be noted that in a Hilbert space, the closure of the range of each operator is automatically orthogonally complemented. In addition, for an adjointable operator T on a Hilbert C^*-module, the range of T is closed if and only if the range of its adjoint operator T^* is closed, and in this case the ranges both of T and of T^* are also automatically orthogonally complemented (see [9, Theorem 3.2]).

On the other hand, recently in [7] Frank and Sharifi obtained some results on the existence of generalized inverses of (densely defined closed) regular module operator as follows. A regular module operator (especially a bounded adjointable module operator) T admits a (possibly unbounded) regular module operator as its generalized inverse if and only if $\overline{R(T^*)}$ is an orthogonal direct summand of the graph of T; This generalized inverse is bounded if and only if additionally $R(T)$ is closed. These results give some more background information on this paper.

We first recall some basic knowledge about Hilbert C^*-modules. Throughout this paper, $\mathcal{A}$ is a C^*-algebra. An inner-product $\mathcal{A}$—module is a linear space E which is a right A-module, together with a map $(x, y) \to \langle x, y\rangle : E \times E \to \mathcal{A}$ such that for any $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$, the following conditions hold:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$;
(iv) $\langle x, x\rangle \geqslant 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An inner-product $\mathcal{A}$-module E which is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is called a (right) Hilbert $\mathcal{A}$-module.

Suppose that E, F are two Hilbert $\mathcal{A}$-modules, let $L_{\mathcal{A}}(E,F)$ be the set of all maps $T : E \to F$ for which there is a map $T^* : F \to E$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \text{for each } x \in E \text{ and } y \in F.$$

It is known that any element T of $L_{\mathcal{A}}(E,F)$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = T(x)a$ for $x \in E$ and $a \in \mathcal{A}$. For any $T \in L_{\mathcal{A}}(E,F)$, the range space and the null space of T are denoted by $R(T)$ and $N(T)$, respectively. We call $L_{\mathcal{A}}(E,F)$ the set of adjointable operators from E to F. We denote by $B_{\mathcal{A}}(E,F)$ the set of all bounded linear $\mathcal{A}$-maps, and therefore $L_{\mathcal{A}}(E,F) \subseteq B_{\mathcal{A}}(E,F)$. In case $E = F, L_{\mathcal{A}}(E)$, to which we abbreviate $L_{\mathcal{A}}(E,F)$, is a C^*-algebra. Moreover we denote by $L_{\mathcal{A}}(E)_{sa}$ and $L_{\mathcal{A}}(E)_+$ the set of hermitian and positive elements of $L_{\mathcal{A}}(E)$, respectively.

We say that a closed submodule E_1 of E is topologically complemented if there is a closed submodule E_2 of E such that $E_1 + E_2 = E, E_1 \cap E_2 = 0$ (briefly, $E = E_1 \widetilde{\oplus} E_2$). If moreover $E_2 = E_1^{\perp}$, where $E_1^{\perp} = \{x \in E : \langle x, y\rangle = 0 \text{ for each } y \in E_1\}$, we say E_1 is orthogonally complemented and briefly denote the sum by $E = E_1 \oplus E_2$.

Compared to the Hilbert space case, there exist some differences when we deal with operators on a Hilbert C^*-module. For instance, a closed topologically complemented submodule may not be orthogonally complemented, meanwhile the fundamental Riesz representation theorem concerning the bounded linear functionals may also be not true. The reader may refer to [8,9,11] for details.

Throughout this paper, E, F and G are Hilbert $\mathcal{A}$-modules.

1. Solutions to the equation $TX = T'$

Lemma 1.1 ([11], Lemma 15.3.5; [9], Theorem 3.2). *Let $T \in L_{\mathcal{A}}(E,F)$, then*

(1) $N(T) = N(|T|), \quad N(T^*) = R(T)^{\perp}, \quad N(T^*)^{\perp} = R(T)^{\perp\perp} \supseteq \overline{R(T)}$.
(2) *$R(T)$ is closed if and only if $R(T^*)$ is closed, and in this case $R(T)$ and $R(T^*)$ are orthogonally complemented with $R(T) = N(T^*)^{\perp}$ and $R(T^*) = N(T)^{\perp}$.*

Theorem 1.1. *Let $T' \in L_{\mathcal{A}}(G,F)$ and $T \in L_{\mathcal{A}}(E,F)$ with $\overline{R(T^*)}$ orthogonally complemented. The following statements are equivalent:*

(i) *$T'T'^* \leqslant \lambda TT^*$ for some $\lambda > 0$;*
(ii) *There exists $\mu > 0$ such that $\|T'^*z\| \leqslant \mu\|T^*z\|$ for all $z \in F$;*
(iii) *There exists $D \in L_{\mathcal{A}}(G,E)$ such that $T' = TD$, i.e., $TX = T'$ has a solution;*
(iv) *$R(T') \subseteq R(T)$.*

Moreover there exists a unique operator D which satisfies the conditions

$$T' = TD, \quad R(D) \subseteq N(T)^{\perp}.$$

In this case,

$$\|D\|^2 = inf\{\lambda : T'T'^* \leqslant \lambda TT^*\} \text{ and } R(D) \subseteq \overline{R(T^*)}; \quad N(D) = N(T'),$$

and this D is called the reduced solution of the equation $TX = T'$.

Proof. By Lemma 1.1, we know there exists an orthogonal decomposition

$$E = \overline{R(T^*)} \oplus N(T).$$

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious.

(iii) $\Rightarrow$ (i): If $TX = T'$ has a solution $D \in L_{\mathcal{A}}(G,E)$, then $T'T'^* = TD^*DT^* \leqslant \|D\|^2TT^*$. We may choose $\lambda = \|D\|^2$.

(ii) $\Rightarrow$ (iii): Let $\mu > 0$ such that $\|T'^*z\| \leqslant \mu\|T^*z\|$ for all $z \in F$. We can define a map

$$D' : R(T^*) \to R(T'^*), \quad T^*z \mapsto T'^*z, \quad \forall z \in F.$$

By the linearity of T^* and T'^*, we know D' is a linear map. Moreover, for any $z \in F$,

$$\|D'(T^*z)\| = \|T'^*z\| \leqslant \mu\|T^*z\|.$$

So we can extend D' linearly and continuously to $\overline{R(T^*)}$, which is denoted still by D' for convenience.

Then we may define the map $\widetilde{D'} : E \to G$ by

$$\widetilde{D'}(y) = \begin{cases} D'(y), & y \in \overline{R(T^*)}; \\ 0, & y \in N(T), \end{cases}$$

and then $\widetilde{D'}T^* = T'^*$.

Let $P : E \to \overline{R(T^*)}$ be the orthogonal projection onto $\overline{R(T^*)}$. If $y_1, y_2 \in E$ with $Ty_1 = Ty_2$, then $y_1 - y_2 \in N(T)$ and so $Py_1 = Py_2$.

Set

$$D : G \to E, \quad x \mapsto PT^{-1}T'x, \quad \forall x \in G,$$

where T^{-1} does not refer to the inverse map of T (since T is not necessarily invertible) but the expression of inverse image. By the discussion in the last paragraph the map D is well defined.

For any $z \in F$ and $x \in G$, we have

$$\begin{aligned} \langle \widetilde{D'}(T^*z), x\rangle &= \langle T'^*z, x\rangle = \langle z, T'x\rangle \\ &= \langle z, TDx\rangle = \langle T^*z, Dx\rangle. \end{aligned}$$

For any $x \in G$ and $y_0 \in N(T)$, we have

$$\langle \widetilde{D'}y_0, x\rangle = 0 = \langle y_0, Dx\rangle.$$

Therefore, applying the orthogonal decomposition $E = \overline{R(T^*)} \oplus N(T)$, we obtain that for arbitrary $y \in E, x \in G$,

$$\langle \widetilde{D'}y, x\rangle = \langle y, Dx\rangle.$$

It follows that $D \in L_{\mathcal{A}}(G, E)$ and $D^* = \widetilde{D'}$. Therefore $(\widetilde{D'})^* = D$ and $TD = T'$.

(iv) $\Rightarrow$ (ii): Since for any $x \in G, T'x \in R(T)$, there exists $y \in N(T)^{\perp} \subseteq E$ such that $T'x = Ty$. Set $D(x) = y$, and then $TDx = T'x, \forall x \in G$.

It is easy to know D is well defined. Indeed, if $y_1, y_2 \in N(T)^{\perp}$ with $T'x = Ty_1 = Ty_2$, then we have $y_1 - y_2 \in N(T) \cap N(T)^{\perp}$, i.e., $y_1 = y_2$.

Moreover, D is linear. For every $x, x' \in G, \alpha, \beta \in \mathbb{C}$, we get

$$T'(\alpha x + \beta x') = \alpha T'x + \beta T'x' = \alpha TDx + \beta TDx' = T(\alpha Dx + \beta Dx').$$

Because $\alpha Dx + \beta Dx' \in N(T)^{\perp}$, by the definition of D, we get $D(\alpha x + \beta x') = \alpha Dx + \beta Dx'$.

Finally D, whose domain is G, is bounded as a linear operator from Banach space G to Banach space E. In fact, let $x_n \to x, Dx_n \to y$ as $n \to \infty$. Since T and T' are continuous, we know

$$TDx_n = T'x_n \to T'x, \quad TDx_n \to Ty, \quad \text{as } n \to \infty.$$

By the uniqueness of limit, we get $T'x = Ty$ and

$$T(y - Dx) = Ty - T'x = 0.$$

Since $y, Dx \in N(T)^{\perp}$, we have $Dx = y$. By virtue of the closed graph theorem about linear operators on Banach space (see [10] 2.15), we know that D is bounded.

For each $x \in G, z \in F$, it follows that

$$\langle Dx, T^*z\rangle = \langle TDx, z\rangle = \langle T'x, z\rangle = \langle x, T'^*z\rangle.$$

Replacing x by T'^*z, we get $\langle DT'^*z, T^*z\rangle = \langle T'^*z, T'^*z\rangle$, and

$$\|T'^*z\|^2 \leqslant \|DT'^*z\|\|T^*z\| \leqslant \|D\|\|T'^*z\|\|T^*z\|.$$

Therefore

$$\|T'^*z\| \leqslant \|D\|\|T^*z\|.$$

Now we have proved the equivalence of (i), (ii), (iii) and (iv).

If there exists another operator $D_1 \in L_{\mathcal{A}}(G,E)$ such that $T' = TD_1$ and $R(D_1) \subseteq N(T)^{\perp}$. For each $x \in G$, $TD(x) = TD_1(x) = T'(x)$ and then $T(D-D_1)(x) = 0$. It follows that $R(D-D_1) \subseteq N(T) \cap N(T)^{\perp} = \{0\}$, and so $D = D_1$. Uniqueness is proved.

If D is the reduced solution, then $D^*(y) = 0$ $(\forall y \in N(T))$, for $R(D) \subseteq N(T)^{\perp}$. We note that

$$\|D\|^2 = \|D^*\|^2 = inf\{c : \|D^*y\|^2 \leqslant c\|y\|^2, \text{ for all } y \in E\}.$$

For any $\lambda > 0$ with $T'T'^* \leqslant \lambda TT^*$, since $TD = T'$,

$$\|D^*T^*z\|^2 = \|T'^*z\|^2 \leqslant \lambda\|T^*z\|^2, \quad \text{for all } z \in F.$$

So we get $\|D\|^2 \leqslant inf\{\lambda : T'T'^* \leqslant \lambda TT^*\}$.

On the other hand, $T'T'^* \leqslant \|D\|^2TT^*$, hence $\|D\|^2 \geqslant inf\{\lambda : T'T'^* \leqslant \lambda TT^*\}$. Thus, we have $\|D\|^2 = inf\{\lambda : T'T'^* \leqslant \lambda TT^*\}$.

Clearly $N(D) \subseteq N(T')$. Since $E = \overline{R(T^*)} \oplus N(T)$ and $R(D) \subseteq N(T)^{\perp}$, we have $R(D) \subseteq \overline{R(T^*)}$. Let $x \in N(T')$, then $\langle Dx, T^*z\rangle = \langle TDx, z\rangle = \langle T'x, z\rangle = 0$ for all $z \in F$, i.e., $Dx \in \overline{R(T^*)}^{\perp}$, and so $Dx = 0$. Therefore $N(T') \subseteq N(D)$, and so $N(T') = N(D)$. □

Remark 1.1. Since every closed subspace of Hilbert space is orthogonally complemented, Theorem 1.1 is a generalization of the classical Douglas theorem from Hilbert space to Hilbert C^*-module.

Remark 1.2. The solution $D \in L_{\mathcal{A}}(G,E)$ of $TX = T'$ we construct in the proof of Theorem 1.1 is determined by its conjugation D^*, which is defined as follows:

$$\begin{cases} D^*(T^*z) = T'^*z, & \text{for any } z \in F; \\ D^*y = 0, & y \in N(T) = R(T^*)^{\perp}. \end{cases}$$

Concretely,

$$Dx = PT^{-1}T'x, \quad \text{for any } x \in G,$$

where $P : E \to \overline{R(T^*)}$ is the orthogonal projection on $\overline{R(T^*)}$, and T^{-1} does not refer to the inverse map of T (since T is not necessarily invertible) but the expression of inverse image. Moreover since $\overline{R(T^*)} \subseteq N(T)^{\perp}$, this D is the reduced resolution of $TX = T'$.

Finally it is easy to see that the general solution of $TX = T'$ is of the form

$$X = D + K,$$

where $K \in L_{\mathcal{A}}(G,E)$ is arbitrary with $R(K) \subseteq N(T)$.

Corollary 1.1. *Let $C \in L_{\mathcal{A}}(G,F), A \in L_{\mathcal{A}}(E,F)$ with $\overline{R(A^*)} \subseteq E$ orthogonally complemented. If there exists an element X in $B_{\mathcal{A}}(G,E)$ such that $AX = C$, then there exists an element X in $L_{\mathcal{A}}(G,E)$ such that $AX = C$.*

Corollary 1.2. *Let $T \in L_{\mathcal{A}}(E,F), T' \in L_{\mathcal{A}}(G,F)$ with $\overline{R(T^*)} \subseteq E$ orthogonally complemented and $R(T') \subseteq R(T)$, and let $Q \in L_{\mathcal{A}}(E)$ be an idempotent such that $N(Q) = N(T)$. Then there exists a unique solution $C \in L_{\mathcal{A}}(G,E)$ of*

$$TX = T', \quad R(X) \subseteq R(Q).$$

Proof. Since Q is an idempotent and $N(Q) = N(T), R(TQ) = R(T)$. By Theorem 1.1 there is a solution D to $TQX = T'$. Then $C = QD$ is a solution to $TX = T'$ and $R(X) \subseteq R(Q)$.

If C_1, C_2 are two solutions to $TX = T'$ and $R(X) \subseteq R(Q)$, then $T(C_1 - C_2) = 0$, and so

$$R(C_1 - C_2) \subseteq N(T) \cap R(Q) = N(Q) \cap R(Q) = 0,$$

i.e., $C_1 = C_2$. This completes the proof of the uniqueness of the solution. □

Corollary 1.3. *Let $T \in L_{\mathcal{A}}(E,F)$ with $\overline{R(T^*)} \subseteq E$ orthogonally complemented and Q be an idempotent in $L_{\mathcal{A}}(F)$ such that $R(QT) \subseteq R(T)$. Then the reduced (i.e., $R(D) \subseteq N(T)^{\perp}$) solution D of $TX = QT$ is an idempotent.*

Proof. We note that

$$TD^2 = QTD = Q^2T = QT,$$

i.e., D^2 is another solution of $TX = QT$. Also,

$$R(D^2) \subseteq R(D) \subseteq N(T)^{\perp}.$$

Thus D^2 is a reduced solution of $TX = QT$. By the proof of the uniqueness of the reduced solution in Theorem 1.1, it must be $D = D^2$. □

Corollary 1.4 ([14, Theorem 2.1]). *Let $T \in L_{\mathcal{A}}(E,F)$ with $\overline{R(T^*)} \subseteq E$ orthogonally complemented (especially if $R(T)$ is closed by Lemma 1.1(2)). Let $Q \in L_{\mathcal{A}}(E)$, $P \in L_{\mathcal{A}}(F)$ be two idempotents such that*

$$R(P) = R(T), \quad N(Q) = N(T).$$

Then there exists a unique solution $C \in L_{\mathcal{A}}(F,E)$ of

$$TX = P, \quad R(X) \subseteq R(Q).$$

Furthermore, $TCT = T$, $CTC = C$ and $CT = Q$.

Proof. By Corollary 1.2 there exists a unique solution C.

Obviously, $TCT = PT = T$.

Since $R(T^* - QT^*) \subseteq N(Q) = N(T)$, $TT^* = TQT^*$. It follows that $T(CTT^* - QT^*) = 0$, and then $R(CTT^* - QT^*) \subseteq R(Q) \cap N(T) = R(Q) \cap N(Q) = \{0\}$. Therefore $CTT^* = QT^*$, i.e., $(CT - Q)|_{R(T^*)} = 0$.

On the other hand, by $N(Q) = N(T)$, it is clear that $(CT - Q)|_{N(T)} = 0$.

By the decomposition $E = \overline{R(T^*)} \oplus N(T)$ and the continuity of $CT - Q$, we get $CT = Q$, and then $CTC = C$ for $R(C) \subseteq R(Q)$. □

Theorem 1.2. *Let $T', T \in L_{\mathcal{A}}(E,F)$ with $\overline{R(T^*)}$ orthogonally complemented. Then $TX = T'$ has a hermitian solution $X \in L_{\mathcal{A}}(E)$ if and only if*

$$R(T') \subseteq R(T), \quad T'T^* \in L_{\mathcal{A}}(F)_{sa}.$$

In this case, $D = D^*$, and the general hermitian solution is of the form $X = D + K$, where D is the reduced solution and $K \in L_{\mathcal{A}}(E)$ is arbitrary self-adjoint operator with $R(K) \subseteq N(T)$.

Proof. Let X be a hermitian solution of $TX = T'$, we have $XT^* = T'^*$, and then

$$T'T^* = TXT^* = TT'^* = (T'T^*)^*.$$

Conversely, assume that $R(T') \subseteq R(T)$, and $T'T^* \in L_{\mathcal{A}}(F)_{sa}$. By Theorem 1.1, there exists a unique reduced solution $D \in L_{\mathcal{A}}(E)$ such that

$$TD = T', \quad Dx = PT^{-1}T'x \quad (\forall x \in E), \quad D^*x = \begin{cases} T'^*y, & x = T^*y, \quad y \in F; \\ 0, & x \in N(T). \end{cases}$$

It follows that for all $y \in F$,

$$T'T^*y = TT'^*y = TD^*T^*y,$$

i.e., $(T' - TD^*)T^*y = 0$. Since $T' - TD^*$ is continuous, we get

$$(T' - TD^*)x = 0, \quad \text{for all } x \in \overline{R(T^*)}.$$

For each $x \in N(T)$,

$$T^*TDx = T^*T'x = T'^*Tx = 0,$$

i.e., $Dx \in N(T^*T) = N(T)$. Therefore $T'x = TDx = 0$, and so $Dx = 0$ for $N(D) = N(T')$ by Theorem 1.1. It follows that

$$(T' - TD^*)x = 0, \quad \text{for all } x \in N(T).$$

Therefore $T'x = TD^*x$ $(\forall x \in E)$, i.e.,

$$T' = TD^*, \quad DT^* = T'^*.$$

It follows that

$$Dx = \begin{cases} T'^*y, & x = T^*y, \quad y \in F; \\ 0, & x \in N(T). \end{cases}$$

Thus we get $D = D^*$. $\square$

Theorem 1.3. *Let $T, T' \in L_{\mathcal{A}}(E, F)$ such that $\overline{R(T^*)}$ orthogonally complemented. Then $TX = T'$ has a positive solution $X \in L_{\mathcal{A}}(E)$ if and only if*

$$R(T') \subseteq R(T), \quad T'T^* \geqslant 0.$$

In this case, $D \geqslant 0$, and the general positive solution is of the form

$$X = D + K,$$

where D is the reduced solution and $K \in L_{\mathcal{A}}(E)$ is arbitrary self-adjoint operator with $R(K) \subseteq N(T)$ and $K \geq -D$. If moreover $R(T'T^*)$ is closed, then $R(T') = R(T'T^*) = R(TT'^*)$.

Proof. Let $X \geqslant 0$ be a positive solution to $TX = T'$. Then $T'T^* = TXT^* \geqslant 0$.

Conversely, since $T'T^* \geqslant 0$, and $R(T') \subseteq R(T)$, by Theorem 1.2 there exists a unique hermitian reduced solution $D \in L_{\mathcal{A}}(E)$ such that

$$TD = T', \quad Dx = D^*x = \begin{cases} T'^*y, & x = T^*y, \quad y \in F; \\ 0, & x \in N(T). \end{cases}$$

For $x \in E$, without generality we assume that $x = T^*y + z$, where $y \in F, z \in N(T)$, it follows that

$$\begin{aligned} \langle Dx, x\rangle &= \langle D(T^*y + z), T^*y + z\rangle = \langle DT^*y, T^*y\rangle \\ &= \langle TDT^*x, y\rangle = \langle T'T^*y, y\rangle \geqslant 0. \end{aligned}$$

Therefore we get $D \geqslant 0$.

In the case that there is a positive solution X to $TX = T'$, we claim that $N(T'^*) = N(TT'^*)$. In fact for $y \in N(TT'^*)$, $TT'^*(y) = TXT^*y = 0$. Hence we get $X^{\frac{1}{2}}T^*y = 0$, and then $T'^*y = XT^*y = 0$.

Therefore, if $R(T'T^*)$ is closed, then

$$R(T') \subseteq N(T'^*)^{\perp} = N(TT'^*)^{\perp} = R(T'T^*) \subseteq R(T'),$$

i.e., $R(T') = R(T'T^*)$. Since $T'T^* \geqslant 0, T'T^* = TT'^*$, and this completes the proof. $\square$

Remark 1.3. From the proof of Theorems 1.2 and 1.3, it is easy to see that if the equation $TX = T'$ has a hermitian (positive) solution, then the reduced solution is just hermitian (positive).

2. Solutions to the equations $TX = T', XS = S'$

In this section we assume $T, T' \in L_{\mathcal{A}}(E, F)$ and $S, S' \in L_{\mathcal{A}}(G, E)$. We will study the common solutions, common hermitian and positive solutions to the following equations:

$$TX = T', \quad XS = S'.$$

This problem has been considered for matrices, Hilbert space operators and Hilbert C^*-module operators with closed ranges in [12,2,13], respectively by use of the generalized inverses. We will discuss this problem for more general Hilbert C^*-module operators and obtain some new results without use of the generalized inverses.

Theorem 2.1. *Suppose that $\overline{R(T^*)}$ and $\overline{R(S)}$ are closed orthogonally complemented submodules of E. Then*

$$TX = T', \quad XS = S'$$

have a common solution $X \in L_{\mathcal{A}}(E)$ if and only if

$$R(T') \subseteq R(T), \quad R(S'^*) \subseteq R(S^*), \quad TS' = T'S.$$

In this case, the general common solution is of the form

$$X = D_1 + D_2^* - PD_2^* + K,$$

where D_1, D_2 are the reduced solutions of $TX = T', S^*X = S'^*$ respectively, P is the projection of E onto $\overline{R(T^*)}$, and K is any operator in $L_{\mathcal{A}}(E)$ with $R(K) \subseteq N(T), R(S) \subseteq N(K)$.

Proof. According to the assumption, there exist two kinds of orthogonal decompositions:

$$E = \overline{R(T^*)} \oplus N(T); \quad E = \overline{R(S)} \oplus N(S^*).$$

Let P be the projection of E onto $\overline{R(T^*)}$.

First we prove the sufficiency. By Theorem 1.1, we know there exist operators $D_1, D_2 \in L_{\mathcal{A}}(E)$ satisfying

$$TD_1 = T', \quad R(D_1) \subseteq N(T)^{\perp};$$
$$D_2^*S = S', \quad R(D_2) \subseteq N(S^*)^{\perp}.$$

In fact, D_1, D_2 are the reduced solutions of equations $TX = T', S^*X = S'^*$, respectively.

Since $R(I-P) \subseteq N(T)$ by the orthogonal decompositions of E, $T(I-P)S' = 0$. Then we have $TPS' = TS' = T'S = TD_1S$, and so $T(PS' - D_1S) = 0$, i.e., $R(PS' - D_1S) \subseteq N(T)$. Since clearly $R(PS' - D_1S) \subseteq N(T)^{\perp}$, we get $PS' = D_1S$.

Therefore, set $X = D_1 + D_2^* - PD_2^*$. It is the common solution as verified as follows:

$$TX = TD_1 + TD_2^* - TPD_2^* = T' + T(I-P)D_2^* = T';$$
$$XS = D_1S + D_2^*S - PD_2^*S = D_1S + S' - PS' = S'.$$

Thus, we complete the proof for sufficiency.

The necessity is obvious. By Theorem 1.1, we have $R(T') \subseteq R(T), R(S'^*) \subseteq R(S^*)$. Moreover, $TS' = TXS = T'S$. $\square$

Lemma 2.1. *Let $U \in L_{\mathcal{A}}(E), V \in L_{\mathcal{A}}(F,E), L \in L_{\mathcal{A}}(F)$. Then $\begin{pmatrix} U & V \\ V^* & L \end{pmatrix} \geqslant 0$ if and only if $U \geqslant 0, L \geqslant 0$, and*

$$\varphi(\langle x, Vy\rangle)\varphi(\langle Vy, x\rangle) \leqslant \varphi(\langle Ux, x\rangle)\varphi(\langle Ly, y\rangle)$$

for any $x \in E, y \in F$ and any state $\varphi \in S(\mathcal{A})$.

Proof. By assumption we know $\begin{pmatrix} U & V \\ V^* & L \end{pmatrix} \in L_{\mathcal{A}}(E \oplus F, E \oplus F)$. For $x \in E, y \in F$,

$$\left\langle \begin{pmatrix} U & V \\ V^* & L \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \langle Ux, x\rangle + \langle Ly, y\rangle + \langle Vy, x\rangle + \langle V^*x, y\rangle.$$

Therefore, $\begin{pmatrix} U & V \\ V^* & L \end{pmatrix} \geqslant 0$, if and only if, for any $x \in E, y \in F$,

$$\langle Ux, x\rangle + \langle Ly, y\rangle + \langle Vy, x\rangle + \langle V^*x, y\rangle \geqslant 0,$$

(and so $U \geqslant 0, L \geqslant 0$) if and only if, for any state $\varphi \in S(\mathcal{A})$, and $x \in E, y \in F$,

$$\varphi(\langle Ux, x\rangle) + \varphi(\langle Ly, y\rangle) + \varphi(\langle Vy, x\rangle) + \varphi(\langle x, Vy\rangle) \geqslant 0,$$

which is to say

$$\varphi(\langle Ux, x\rangle) + \varphi(\langle Ly, y\rangle) + 2\mathrm{Re}(\varphi(\langle Vy, x\rangle)) \geqslant 0.$$

As in the standard proof of the Cauchy–Schwarz inequality, replace y by $\lambda e^{i\theta} y$ in the last inequality so as to make $\varphi(\langle Vy, x\rangle)$ real, where $\lambda \in \mathbb{R}, \theta \in [0, 2\pi]$. Then the left-hand of the last inequality is a quadratic form in λ, and applying δ-discriminant we get

$$|\varphi(\langle Vy, x\rangle)|^2 \leqslant \varphi(\langle Ux, x\rangle)\, \varphi(\langle Ly, y\rangle),$$

i.e.,

$$\varphi(\langle x, Vy\rangle)\varphi(\langle Vy, x\rangle) \leqslant \varphi(\langle Ux, x\rangle)\varphi(\langle Ly, y\rangle).$$

Conversely if $U \geqslant 0, L \geqslant 0$, and $\varphi(\langle x, Vy\rangle)\varphi(\langle Vy, x\rangle) \leqslant \varphi(\langle Ux, x\rangle)\varphi(\langle Ly, y\rangle)$ for some $x \in E, y \in F$ and $\varphi \in S(\mathcal{A})$, then

$$\begin{aligned} &\varphi(\langle Ux, x\rangle) + \varphi(\langle Ly, y\rangle) + 2\mathrm{Re}(\varphi(\langle Vy, x\rangle)) \\ &\geqslant \varphi(\langle Ux, x\rangle) + \varphi(\langle Ly, y\rangle) - 2|\varphi(\langle Vy, x\rangle)| \\ &\geqslant \varphi(\langle Ux, x\rangle) + \varphi(\langle Ly, y\rangle) - 2\sqrt{\varphi(\langle Ux, x\rangle)}\sqrt{\varphi(\langle Ly, y\rangle)} \qquad \square \\ &= (\sqrt{\varphi(\langle Ux, x\rangle)} - \sqrt{\varphi(\langle Ly, y\rangle)})^2 \\ &\geqslant 0. \end{aligned}$$

Corollary 2.1. *Suppose that* $\overline{R(T^*) + R(S)}$ *is a closed orthogonally complemented submodule of* E. *Then the following statements are equivalent*:

(1) $TX = T', S^*X = S'^*$ *have a common solution* $X \in L_{\mathcal{A}}(E)$;
(2) $\{(T'x, S'^*x) : x \in E\}$ *is a subset of* $\{(Tx, S^*x) : x \in E\} \subseteq F \oplus G$;
(3) *There exists* $\mu > 0$ *such that* $\|T'^*y + S'z\| \leqslant \mu\|T^*y + Sz\|$ *for any* $y \in F$ *and* $z \in G$;
(4) *There exists* $\lambda > 0$ *such that* $\lambda S^*S - S'^*S' \geqslant 0, \lambda TT^* - T'T'^* \geqslant 0$, *and*

$$\begin{aligned} &\varphi(\langle y, (\lambda TS - T'S')z\rangle)\, \varphi(\langle(\lambda TS - T'S')z, y\rangle) \\ &\quad \leqslant \varphi(\langle(\lambda TT^* - T'T'^*)y, y\rangle)\varphi(\langle(\lambda S^*S - S'^*S')z, z\rangle) \end{aligned}$$

for any $y \in F, z \in G$ *and any state* $\varphi \in S(\mathcal{A})$.

Proof. It is known that $TX = T', S^*X = S'^*$ have a common solution $X \in L_{\mathcal{A}}(E)$ if and only if $\begin{pmatrix} T \\ S^* \end{pmatrix} X = \begin{pmatrix} T' \\ S'^* \end{pmatrix}$ has a solution, where $\begin{pmatrix} T \\ S^* \end{pmatrix}, \begin{pmatrix} T' \\ S'^* \end{pmatrix} \in L_{\mathcal{A}}(E, F \oplus G)$.

By assumption, $\overline{R\left(\begin{pmatrix} T \\ S^* \end{pmatrix}^*\right)} = \overline{R(T^*) + R(S)} \subseteq E$ is a closed orthogonally complemented submodule of E.

By Theorem 1.1 we know that $TX = T', S^*X = S'^*$ have a common solution $X \in L_{\mathcal{A}}(E)$; if and only if, $R\left(\begin{pmatrix} T' \\ S'^* \end{pmatrix}\right) \subseteq R\left(\begin{pmatrix} T \\ S^* \end{pmatrix}\right)$, i.e., $\{(T'x, S'^*x) : x \in E\}$ is a subset of $\{(Tx, S^*x) : x \in E\} \subseteq F \oplus G$; if and only if, there exists $\mu > 0$ such that $\|T'^*y + S'z\| \leqslant \mu\|T^*y + Sz\|$ for any $y \in F$ and $z \in G$; if and only if, there exists $\lambda > 0$ such that

$$\begin{pmatrix} T' \\ S'^* \end{pmatrix}\begin{pmatrix} T' \\ S'^* \end{pmatrix}^* \leqslant \lambda \begin{pmatrix} T \\ S^* \end{pmatrix}\begin{pmatrix} T \\ S^* \end{pmatrix}^*.$$

It is easy to verify that

$$\lambda \begin{pmatrix} T \\ S^* \end{pmatrix} \begin{pmatrix} T \\ S^* \end{pmatrix}^* - \begin{pmatrix} T' \\ S'^* \end{pmatrix} \begin{pmatrix} T' \\ S'^* \end{pmatrix}^* = \begin{pmatrix} \lambda TT^* - T'T'^* & \lambda TS - T'S' \\ \lambda S^*T^* - S'^*T'^* & \lambda S^*S - S'^*S' \end{pmatrix},$$

which is denoted by Δ.

By Lemma 2.1, $\Delta \geqslant 0$ if and only if $\lambda TT^* - T'T'^* \geqslant 0, \quad \lambda S^*S - S'^*S' \geqslant 0$, and for any $y \in F, z \in G$ and any state $\varphi \in S(\mathcal{A})$,

$$\begin{aligned} &\varphi(\langle y, (\lambda TS - T'S')z\rangle)\, \varphi(\langle(\lambda TS - T'S')z, y\rangle) \\ &\quad \leqslant \varphi(\langle(\lambda TT^* - T'T'^*)y, y\rangle)\, \varphi(\langle(\lambda S^*S - S'^*S')z, z\rangle). \end{aligned}$$

This completes the proof. $\square$

Proposition 2.1. *Suppose that $\overline{R(T^*) + R(S)}$ is a closed orthogonally complemented submodule of E. Then*

$$TX = T', \quad XS = S'$$

have a common hermitian solution $X \in L_{\mathcal{A}}(E)$, if and only if,

$$TS' = T'S, \quad T'T^* \text{ and } S'^*S \text{ are hermitian},$$

and one of the following equivalent conditions holds:

(1) *$\{(T'x, S'^*x) : x \in E\}$ is a subset of $\{(Tx, S^*x) : x \in E\} \subseteq F \oplus G$;*
(2) *There exists $\mu > 0$ such that $\|T'^*y + S'z\| \leqslant \mu\|T^*y + Sz\|$ for any $y \in F$ and $z \in G$;*
(3) *There exists $\lambda > 0$ such that $\lambda S^*S - S'^*S' \geqslant 0, \lambda TT^* - T'T'^* \geqslant 0$, and*

$$\begin{aligned} &\varphi(\langle y, (\lambda TS - T'S')z\rangle)\, \varphi(\langle(\lambda TS - T'S')z, y\rangle) \\ &\quad \leqslant \varphi(\langle(\lambda TT^* - T'T'^*)y, y\rangle)\, \varphi(\langle(\lambda S^*S - S'^*S')z, z\rangle) \end{aligned}$$

for any $y \in F, z \in G$ and any state $\varphi \in S(\mathcal{A})$.

Proof. $TX = T', XS = S'$ have a common hermitian solution if and only if $\begin{pmatrix} T \\ S^* \end{pmatrix} X = \begin{pmatrix} T' \\ S'^* \end{pmatrix}$ has a hermitian solution, and if and only if, $\begin{pmatrix} T' \\ S'^* \end{pmatrix} \begin{pmatrix} T \\ S^* \end{pmatrix}^*$ is hermitian and $R\left(\begin{pmatrix} T' \\ S'^* \end{pmatrix}\right) \subseteq R\left(\begin{pmatrix} T \\ S^* \end{pmatrix}\right)$ by Theorem 1.2. It is easy to see

$$\begin{pmatrix} T' \\ S'^* \end{pmatrix} \begin{pmatrix} T \\ S^* \end{pmatrix}^* = \begin{pmatrix} T'T^* & T'S \\ S'^*T^* & S'^*S \end{pmatrix}; \quad \begin{pmatrix} T \\ S^* \end{pmatrix} \begin{pmatrix} T' \\ S'^* \end{pmatrix}^* = \begin{pmatrix} TT'^* & TS' \\ S^*T'^* & S^*S' \end{pmatrix}.$$

Then $\begin{pmatrix} T' \\ S'^* \end{pmatrix} \begin{pmatrix} T \\ S^* \end{pmatrix}^*$ is hermitian if and only if $T'T^*, S'^*S$ are hermitian and $T'S = TS'$.

From both Theorem 1.1 and Corollary 2.1, we complete the proof of this proposition. $\square$

To simplify the expression, in the following discussion we set

$$W = \begin{pmatrix} T' \\ S'^* \end{pmatrix} \begin{pmatrix} T \\ S^* \end{pmatrix}^* = \begin{pmatrix} T'T^* & T'S \\ S'^*T^* & S'^*S \end{pmatrix}.$$

The next proposition gives some necessary and sufficient conditions for the existence of common positive solutions to the equations $TX = T', XS = S'$.

Proposition 2.2. *Suppose that $\overline{R(T^*) + R(S)}$ is a closed orthogonally complemented submodule of E. Then*

$$TX = T', \quad XS = S'$$

have a common positive solution $X \in L_{\mathcal{A}}(E)$, if and only if $W \geqslant 0$ and one of the following equivalent conditions holds:

(1) *$\{(T'x, S'^*x) : x \in E\}$ is a subset of $\{(Tx, S^*x) : x \in E\} \subseteq F \oplus G$;*
(2) *There exists $\mu > 0$ such that $\|T'^*y + S'z\| \leqslant \mu\|T^*y + Sz\|$ for any $y \in F$ and $z \in G$;*
(3) *There exists $\lambda > 0$ such that $\lambda S^*S - S'^*S' \geqslant 0$, $\lambda TT^* - T'T'^* \geqslant 0$, and*

$$\varphi(\langle y, (\lambda TS - T'S')z\rangle)\,\varphi(\langle(\lambda TS - T'S')z, y\rangle) \leqslant \varphi(\langle(\lambda TT^* - T'T'^*)y, y\rangle)\varphi(\langle(\lambda S^*S - S'^*S')z, z\rangle)$$

for any $y \in F, z \in G$ and any state $\varphi \in S(\mathcal{A})$.

Proof. $TX = T', XS = S'$ have a common positive solution, if and only if,

$$\begin{pmatrix} T \\ S^* \end{pmatrix} X = \begin{pmatrix} T' \\ S'^* \end{pmatrix}$$

has a positive solution, which is equivalent to $W \geqslant 0$ and $R\left(\begin{pmatrix} T' \\ S'^* \end{pmatrix}\right) \subseteq R\left(\begin{pmatrix} T \\ S^* \end{pmatrix}\right)$ by Theorem 1.3.

From both Theorem 1.1 and Corollary 2.1, we complete the proof of this proposition. □

It is easy to see that $W \geqslant 0$ if and only if

$$T'T^* \geqslant 0, \quad S'^*S \geqslant 0, \quad T'S = TS',$$

and for any $y \in F, z \in G$ and any state $\varphi \in S(\mathcal{A})$,

$$\varphi(\langle y, T'Sz\rangle)\varphi(\langle T'Sz, y\rangle) \leqslant \varphi(\langle T'T^*y, y\rangle)\varphi(\langle S'^*Sz, z\rangle).$$

Therefore we have the following Corollary:

Corollary 2.2. *Suppose that $\overline{R(T^*) + R(S)}$ is a closed orthogonally complemented submodule of E. Then*

$$TX = T', \quad XS = S'$$

have a common positive solution $X \in L_{\mathcal{A}}(E)$, if and only if,

$$T'T^* \geqslant 0, \quad S'^*S \geqslant 0, \quad T'S = TS',$$
$$\varphi(\langle y, T'Sz\rangle)\varphi(\langle T'Sz, y\rangle) \leqslant \varphi(\langle T'T^*y, y\rangle)\varphi(\langle S'^*Sz, z\rangle)$$

for any $y \in F, z \in G$ and any state $\varphi \in S(\mathcal{A})$, and one of the following equivalent conditions holds:

(1) *$\{(T'x, S'^*x) : x \in E\}$ is a subset of $\{(Tx, S^*x) : x \in E\} \subseteq F \oplus G$;*
(2) *There exists $\mu > 0$ such that $\|T'^*y + S'z\| \leqslant \mu\|T^*y + Sz\|$ for any $y \in F$ and $z \in G$;*
(3) *There exists $\lambda > 0$ such that $\lambda S^*S - S'^*S' \geqslant 0$, $\lambda TT^* - T'T'^* \geqslant 0$, and*

$$\varphi(\langle y, (\lambda TS - T'S')z\rangle)\,\varphi(\langle(\lambda TS - T'S')z, y\rangle) \leqslant \varphi(\langle(\lambda TT^* - T'T'^*)y, y\rangle)\,\varphi(\langle(\lambda S^*S - S'^*S')z, z\rangle)$$

for any $y \in F, z \in G$ and any state $\varphi \in S(\mathcal{A})$.

Acknowledgement

The authors would like to thank the referee for his/her useful comments and suggestions.

References

[1] D.S. Cvetković-Ilić, A. Dajić, J.J. Koliha, Positive and real-positive solutions to the equation $axa^* = c$ in C^*-algebras, Linear and Multilinear Algebra 55 (2007) 535–543.
[2] A. Dajić, J.J. Koliha, Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators, J. Math. Anal. Appl. 333 (2007) 567–576.

[3] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–416.
[4] X. Fang, The represention of topological groupoid, Acta Math. Sinica 39 (1996) 6–15.
[5] X. Fang, The induced representation of C^*-groupoid dynamical system, Chinese Ann. Math. (B) 17 (1996) 103–114.
[6] X. Fang, The realization of multiplier Hilbert bimodule on bidule space and Tietze extension theorem, Chinese Ann. Math. (B) 21 (2000) 375–380.
[7] M. Frank, K. Sharifi, Adjointability of densely defined closed operators and the Magajna–Schweizer theorem, J. Operator Theory, in press, <arXiv: 0705.2576v2>.
[8] K.K. Jensen, K. Thomsen, Elements of KK-Theory, Birkhauser, Boston, 1991.
[9] E.C. Lance, Hilbert C^*-Modules: A Toolkit for Operator Algebraists, Cambridge University Press, Cambridge, 1995.
[10] W. Rudin, Functional Analysis, McGraw-Hill, Inc., New York, 1991.
[11] N.E. Wegge-Olsen, K-Theory and C^*-Algebras: A Friendly Approach, Oxford University Press, Oxford, 1993.
[12] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computations, Science Press, Beijing/New York, 2004.
[13] Q. Xu, Common hermitian positive solutions to the adjointable operator equations $AX = C, XB = D$, Linear Algebra Appl. 429 (2008) 1–11.
[14] Q. Xu, L. Sheng, Positive semi-definite matrices of adjointable operators on Hilbert C^*-modules, Linear Algebra Appl. 428 (2008) 992–1000.